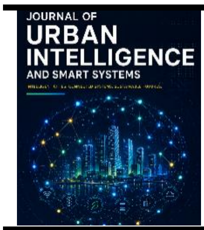

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A Novel Spherical Fuzzy Schweizer-Sklar Prioritized Framework for Resilient Internet Router Selection in Smart City Digital Infrastructures

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ABSTRACT

The digital backbone of any smart city rests upon reliable routing infrastructure, where router reliability governs data throughput for traffic management, public safety, and utility grids. Selecting optimal routers involves intricate trade-offs among bandwidth, security, latency, and cost, all shrouded in expert ambiguity. To methodically resolve this inherent uncertainty, we propose a novel and robust multi-attribute decision-making paradigm anchored in spherical fuzzy sets (SFSs), which capture three-dimensional membership, abstinence, and non-membership grades, seamlessly integrated with Schweizer-Sklar t-norms and t-conorms for flexible parameter-sensitive aggregation. We formally introduce two pioneering operators, namely spherical fuzzy Schweizer-Sklar weighted prioritized aggregation (SFSSWPA) and geometric (SFSSWPG), embedding prioritized hierarchical inter-criteria relationships, thereby ensuring that dominant attributes receive proportionally greater decision weight. The complete multi attribute decision making (MADM) algorithm is validated through a real-world router procurement case study situated within urban critical network infrastructure deployments. Exhaustive comparative assessments against several established fuzzy MADM benchmarks consistently confirm our approach's superior ranking stability and remarkable parameter adaptability.

1. Introduction

The digital backbone of any smart city rests upon reliable routing infrastructure, where router reliability governs data throughput for traffic management, public safety, and utility grids. Selecting optimal routers involves intricate trade-offs among bandwidth, security, latency, and cost, all shrouded in expert ambiguity. Traditional multi attribute decision making (MADM) techniques neglect prioritized relationships and struggle with multifaceted uncertainty. This paper introduces a fully robust paradigm to tackle these complexities, filling a critical void in ever-evolving urban network planning and procurement governance.

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MADM is one of the most emerging research avenues and a widely used technique for data aggregation. It estimates the best alternative(s) from a range of available resources. The concept of MADM is applied in many fields, including artificial intelligence and machine learning. Assessing or choosing the optimal form that maximizes the limited choice while accounting for the preference values provided by experts or decision-makers based on their attribute values is the primary impact of MADM problems. Assigning a numerical value to each desire in real-world difficulties is uncomfortable and challenging for specialists due to unclear, complex decision-making contexts. Considering the preceding puzzle, Zadeh [1] invented the fuzzy set (FS) notion and improved or modified the concept of classical information. In FS theory, a membership degree ($\mathcal{MD}$), restricted to [0,1], is assigned to each object, indicating the degree of belongingness of the object to the set. However, FSs cannot yield satisfactory results when knowledge of $\mathcal{MD}$ is limited. Later, Atanassov [2] produced the paradigm of the intuitionistic fuzzy set (IFS) by incorporating the MD and non-membership degree ($\mathcal{NMD}$) such that $0 \leq \mathcal{MD} + \mathcal{NMD} \leq 1$. To consider the circumstances where $\mathcal{MD} + \mathcal{NMD} \geq 1$, Yager [3] put forward the notion of the Pythagorean fuzzy set (PyFS) by extending the wider region of uncertainties with the constraint $0 \leq \mathcal{MD}^2 + \mathcal{NMD}^2 \leq 1$.

Although FS, IFS, and PyFS are widely utilized in decision-analytic procedures across numerous real-world scenarios, their applicability remains constrained to issues where decisions are articulated in binary terms (yes or no) or combinations thereof. Yet, numerous practical situations appear where these frameworks cannot adequately characterize the entire spectrum of decision-making complexities. For instance, in electoral voting, a voter may have three distinct choices: voting in favor, voting against, or abstaining. Conventional FS, IFS, and PyFS frameworks are not natively equipped to accommodate such multi-faceted decision-making scenarios, highlighting the requirement for more advanced approaches capable of effectively addressing these intricacies. Consequently, Cuong [4] proposed a picture fuzzy set (PFS) as a generalization of available approaches. A PFS consists of three components, namely membership degree $\mu(x)$, degree of abstention $\nu(x)$ and non-membership degree $\eta(x)$ with condition $0 \leq \mu(x) + \nu(x) + \eta(x) \leq 1$. The PFS model has been applied effectively in numerous areas, but it cannot work in certain circumstances. For example, $\mu(x) = 0.4$, $\nu(x) = 0.3$, and $\eta(x) = 0.6$, then we can notice that their sum does not lie in [0,1]. In this regard, the idea of SFS were created by Mahmood *et al.,* [5] under the constraint that $0 \leq \mu^2(x) + \nu^2(x) + \eta^2(x) \leq 1$. The SFS displays a broader domain of applicability compared to IFS, PyFS, and PFS, rendering it a more generic variant of the conventional FS framework. Its distinct structural properties and enhanced decision-making aptitudes have garnered significant interest among researchers, positioning SFS as a promising paradigm for addressing complex uncertainty modelling and decision analysis challenges. Rafiq *et al.,* [6] studied several cosine similarity measures of SFSs and their applications in decision-making. Several authors [7-8] have proposed MADM problems in the environment of SFSs. Ali [9] introduced an innovative score function combining the characteristics of SFSs and projected a MARCOS for MADM problems. The authors of [10] extended the CODAS method within the SFS context, enhancing its applicability in decision-making situations. Mathew *et al.,* [11] unified the AHP and TOPSIS schemes to facilitate system selection under the SFS environment.

1.1 Literature Review of Aggregation Operators

Information aggregation operators (AOs) have appeared as a critical research focus, serving as a vital tool for addressing MADM issues across various fuzzy settings. Their ability to successfully aggregate uncertain and vague information has made them an indispensable constituent of decision-making models. Subsequently, AOs have garnered growing interest from a broad spectrum of scholars, leading to extensive research and advancements in this domain. Hussain *et al.,* [12] developed the theory of Aczel-Alsina AOs under a Pythagorean fuzzy rough environment for medical diagnosis problems. Garg *et al.,* [13] provided a class of Hamacher AOs under an intuitionistic fuzzy setting with decision analytic applications. Ashraf *et al.,* [14] proposed AOs based on Dombi operations and provided detailed proof of their fundamental features. The authors of [15] fostered a class of novel methodologies by employing the characteristics of Einstein AOs by fusing SF information. Zhang *et al.*, [16] discussed an innovative approach within the framework of SFSs, developing an efficient aggregation model capable of handling unknown weight information.

From the existing body of research, it becomes evident that AOs employed across numerous FS variants primarily utilize algebraic, Hamacher, Aczel-Alsina, Frank, and Dombi operational laws. These laws are specific cases of Archimedean t-norms and t-conorms, which serve as generalized extensions of several t-norm and t-conorm families. The Archimedean framework includes a diverse range of special cases, effectively modelling the union and intersection operations within the SFS context. Among these, Schweizer-Sklar operations stand out as notable special cases of Archimedean t-norms and t-conorms, distinguished by their incorporation of a variable parameter. This flexibility improves their versatility, making them superior to other conventional operational methods in this realm. The flexibility of Schweizer-Sklar operations allows decision experts to fine-tune the parameters as required, thereby extending their applicability to a broader range of decision analytic scenarios. In recent eras, scholars have extensively explored Schweizer-Sklar aggregation tools due to their inherent flexibility and versatility in various fuzzy circumstances with decision-making dilemmas. A fresh approach to the interval-valued intuitionistic fuzzy information decision-making system based on Schweizer-Sklar AOs was developed by Zhang [17]. Additionally, as noted in references [18-20] we examined several of the resilient properties and characteristics of the Schweizer-Sklar aggregation tools across a variety of fuzzy settings. The authors of [21] proposed a decision-making approach via Schweizer-Sklar generalized power AOs. The authors of [22] utilized the Pythagorean fuzzy rough set to construct a decision analytic scheme based on the Schweizer-Sklar t-norm. Khan *et al.,* [23] categorized a class of novel Schweizer-Sklar power Heronian mean AOs for SF information.

In MADM, attributes often vary in importance, demanding a structured method to account for their priority relations. To address this complication, Yager [24] introduced the prioritized AOs (PAOs), which systematically integrate the hierarchical significance of attributes into the decision-making process. This operator ensures that higher-priority attributes exert greater influence on the final decision before lower-priority attributes are considered. By integrating priority relationships into the aggregation fusion, the PAOs boost the robustness and reliability of multi-attribute assessments, making them particularly useful in multifaceted decision-making situations. Recently, many scholars have described a wide range of PAOs across various FS variants. Yu and Xu [25] leverage PAO features for the IFS. Sarfraz *et al.,* [26] discuss the concept of IFS-based PAOs. Ullah *et al.,* [27] introduced the concept of prioritized Aczel-Alsina AOs using the complex IFS. Gao [28] further extended the PAO paradigm by integrating the algebraic sum and product in the context of PyFSs. Gul *et al.,* [29] introduced a MADM model based on Schweizer-Sklar PAOs in an intuitionistic fuzzy rough environment. Yang and Zhang [30] developed a MULTIMOORA-based Schweizer-Sklar AOS in a Pythagorean fuzzy environment. Hwang *et al.*, [31] introduced the concept of wireless networks using FS theory based on clustering and stability. Mir *et al.,* [32] developed a theory of internet routing using an improved algorithm that incorporates chaos theory and fuzzy logic. Jutila and Frantti [33] expanded the theory of fuzzy routing by introducing the concept of wireless routers, building on fuzzy flow control. Rahmani *et al.,* [34] presented a concept for routers in heterogeneous networks based on FS analysis and the design of an AQM controller. Naeem *et al.*, [35] presented applications in internet routing and transport network flow using intuitionistic fuzzy graphs. Polshchykov *et al.*, [36] developed the concept of wireless networks in techno-sphere safety control systems. Serik *et al.,* [37] presented a web-based application using OWASP intellectual vulnerability scanners. Rani *et al.*, [38] gave the concept of networking technologies based on congestion-aware secure route selection in ICN. Ali *et al.*, [39] proposed a wireless network that uses a fuzzy-logic-based stable election protocol for cluster head selection.

1.2 Knowledge Gaps and Motivations

When addressing SF information in the context of MADM, several inherent shortcomings arise, serving as the primary motivation for this script:

(1). The SFS model has attracted the most interest among the many extensions of SFs due to its capacity to accomplish conflicting information. The primary benefit of SFSs lies in their ability to integrate AG alongside MG and NMG, encompassing a wider range of uncertain perspectives. Moreover, SFSs empower decision experts to express their judgments with greater independence, enhancing both the precision and dependability of the resulting decisions.

(2). Prioritization plays a fundamental role across various domains of life, as preference inherently influences human decision-making processes. In the context of aggregation procedures, scholars must recognize the critical impact of prioritization in ensuring meaningful and accurate outcomes. However, most of the AOs in the paradigms of SFs cannot consider the degree of prioritization among different attributes, limiting their practical efficiency in various circumstances.

(3). Within the SF decision-making framework, various AOs have been developed to enhance aggregation fusion, often leveraging different t-norms and t-conorms. Notably, the inherent flexibility of SS-TN and SS-TCN grants Schweizer-Sklar-based AOs superior adaptability and robustness compared to operators based on other t-norm families, thereby improving the effectiveness of the aggregation process. To the best of our knowledge, no prior research has integrated PAOs with Schweizer-Sklar operators within the SF context. Therefore, the development of Schweizer-Sklar-based PAOs is both novel and necessary. To bridge this knowledge gap, in this work we have established some Schweizer-Sklar PAOs within the realm of the SF context and explored their desirable characteristics.

(4). The chief advantage of the projected AOs stems from their parametric formulation, which incorporates an intrinsic parameter, σ. This parameter improves both the robustness and adaptability of the aggregation procedure, making the proposed AOs highly effective for real-world MADM problems.

1.3 Main Contributions

The major contributions of this paper are briefed in the following points.

- To incorporate priority levels among input arguments, this study integrates PAOs with Schweizer-Sklar operational laws within the SF framework. This integration leads to the development of novel AOs, namely the SFSSWPA and SFSSWPG operators, which enhances the effectiveness and flexibility of the aggregation process.

- To thoroughly examine the properties of the projected operators, including idempotency, monotonicity, and boundedness.
- This study aims to develop an innovative MADM framework by integrating the proposed AOs within the SF context.
- To illustrate the effectiveness and superiority of the proposed model, a sustainable real-life illustration is conducted.
- Additionally, a comprehensive comparison and sensitivity analysis are conducted to validate the stability and robustness of the derived outcomes.

1.4 Organization of this Study

The rest of this script is outlined as follows: Section 2 introduces cardinal terminologies connected to SFSs, Schweizer-Sklar operations, and prioritized operators. Section 3 formally formulates the SF Schweizer-Sklar weighted prioritized averaging and geometric operators and scrutinizes their key properties. In Section 4, the projected AOs are integrated into a MADM context, utilizing SFSs as representative values. Section 5 presents a case study to demonstrate the practical applicability and effectiveness of the proposed scheme. Section 6 conducts a sensitivity analysis regarding the Schweizer-Sklar parameter and presents a comparative study of numerous existing approaches to validate the effectiveness and adequacy of the proposed method. Lastly, Section 7 presents the study's concluding remarks and future research perspectives.

2. Background Knowledge

Our methodological architecture uniquely synergizes spherical fuzzy sets, Schweizer-Sklar t-norms, and prioritized aggregation. Traditional fuzzy systems ignore abstinence grades and rigidly apply fixed operations. We incorporate the Schweizer-Sklar parameter to dynamically model criterion interactions, capturing nuanced urban stakeholder preferences. The prioritized operators ensure hierarchically superior criteria dominate subordinate ones. This resulting novel framework departs from crisp numerical scoring, embracing interval-valued preferences common in public procurement and tenders, thus perfectly suiting complex urban digital infrastructure decision-making.

In this segment, we briefly review SFSs, SS-TN, SS-TCN, and PAOs.

Definition 1[2]**:** An SFS $\mathcal{S}$ on the universal set $\mho$ is described by:

$$\mathcal{S} = \{(\mu(x), \nu(x), \eta(x)) : x \in \mho\}, \quad (1)$$

where $\mu(x), \nu(x), \eta(x) \in [0,1]$ serve as the MG, AG, and NMG of the element x in the set $\mathcal{S}$, respectively, with the condition: $0 \le \mu^2(x) + \nu^2(x) + \eta^2(x) \le 1$. Additionally, we defined the degree of refusal as:

$$\mathcal{H}(x) = \sqrt{1 - (\mu^2(x) + \nu^2(x) + \eta^2(x))}. \quad (2)$$

For the sake of simplicity, $(\mu(x), \nu(x), \eta(x))$ is titled an SF value (SFV), denoted as $\mathcal{S} = (\mu, \nu, \eta)$.

Definition 2 [40]**:** Let $\mathcal{S} = (\mu, \nu, \eta)$ be an SFV; the score and accuracy functions are respectively articulated as:

$$\mathfrak{S}(\mathcal{S}) = \frac{(2 + \mu^2 - \nu^2 - \eta^2)}{3}, \quad (3)$$

$$\mathfrak{A}(\mathcal{S}) = \frac{(2+\mu^2+\nu^2+\eta^2)}{3}, \quad (4)$$

where, $\mathfrak{S}(\mathcal{S}) \in [-1, 1]$ and $\mathfrak{A}(\mathcal{S}) \in [0, 1]$.
Moreover, for two SFVs $\mathcal{S}_j = (\mu_j, \nu_j, \eta_j);\ j = 1,2$, the ordered relations are described as follows:

1. If $\mathfrak{S}(\mathcal{S}_1) < \mathfrak{S}(\mathcal{S}_2)$, then $\mathcal{S}_1 < \mathcal{S}_2$.
2. If $\mathfrak{S}(\mathcal{S}_1) \succcurlyeq \mathfrak{S}(\mathcal{S}_2)$, then $\mathcal{S}_1 \succcurlyeq \mathcal{S}_2$.
3. If $\mathfrak{S}(\mathcal{S}_1) = \mathfrak{S}(\mathcal{S}_2)$, then
   - If $\mathfrak{A}(\mathcal{S}_1) < \mathfrak{A}(\mathcal{S}_2)$, then $\mathcal{S}_1 < \mathcal{S}_2$.
   - If $\mathfrak{A}(\mathcal{S}_1) \succcurlyeq \mathfrak{A}(\mathcal{S}_2)$, then $\mathcal{S}_1 \succcurlyeq \mathcal{S}_2$.
   - If $\mathfrak{A}(\mathcal{S}_1) = \mathfrak{A}(\mathcal{S}_2)$, then then $\mathcal{S}_1 = \mathcal{S}_2$.

Definition 3: [41] Let $p, q \in [0, 1]$. Then, the SS-TN and SS-TCN among p and q are respectively postulated as:

$$\mathcal{T}_{SS}(p, q) = (p^\sigma + q^\sigma - 1)^{1/\sigma}, \tag{5}$$

$$\mathcal{T}^*{}_{SS}(p, q) = 1 - ((1 - p)^\sigma + (1 - q)^\sigma - 1)^{1/\sigma}, \tag{6}$$

where $\sigma < 0$. Moreover, if $\sigma = 0$, SS-TN and SS-TCN are converted into the algebraic t-norm and t-conorm, respectively.

Definition 4 [42]: Let $\mathcal{S}_j = (\mu_j, \nu_j, \eta_j); j = 1, 2$, be two SFVs and $\theta \succcurlyeq 0$ be any real number. Then, some necessary Schweizer-Sklar operations are expressed as follows:

$$(1).\ \mathcal{S}_1 \oplus \mathcal{S}_2 = \begin{pmatrix} \sqrt{(1 - ((1 - \mu_1^2)^\sigma + (1 - \mu_2^2)^\sigma) - 1)^{\frac{1}{\sigma}}}, \\ \sqrt{((\nu_1^2)^\sigma + (\nu_2^2)^\sigma - 1)^{\frac{1}{\sigma}}}, \\ \sqrt{((\eta_1^2)^\sigma + (\eta_2^2)^\sigma - 1)^{\frac{1}{\sigma}}} \end{pmatrix};$$

$$(2).\ \mathcal{S}_1 \otimes \mathcal{S}_2 = \begin{pmatrix} \sqrt{((\nu_1^2)^\sigma + (\nu_2^2)^\sigma - 1)^{\frac{1}{\sigma}}}, \\ \sqrt{(1 - ((1 - \eta_1^2)^\sigma + (1 - \eta_2^2)^\sigma) - 1)^{\frac{1}{\sigma}}} \\ \sqrt{(1 - ((1 - \mu_1^2)^\sigma + (1 - \mu_2^2)^\sigma) - 1)^{\frac{1}{\sigma}}} \end{pmatrix};$$

$$(3).\ \theta\mathcal{S}_1 = \begin{pmatrix} \sqrt{\left(1 - \left(\theta(1 - \mu_1^2)^\sigma - (\theta - 1)\right)\right)^{\frac{1}{\sigma}}}, \\ \sqrt{\left(\theta(1 - \nu_1^2)^\sigma - (\theta - 1)\right)^{\frac{1}{\sigma}}}, \\ \sqrt{\left(\theta(1 - \eta_1^2)^\sigma - (\theta - 1)\right)^{\frac{1}{\sigma}}} \end{pmatrix};$$

$$(4).\ \mathcal{S}_1{}^\theta = \begin{pmatrix} \sqrt{\left(\theta(1 - \eta_1^2)^\sigma - (\theta - 1)\right)^{\frac{1}{\sigma}}}, \\ \sqrt{1 - \left(\theta(1 - \nu_1^2)^\sigma - (\theta - 1)\right)^{\frac{1}{\sigma}}}, \\ \sqrt{\left(1 - \left(\theta(1 - \mu_1^2)^\sigma - (\theta - 1)\right)\right)^{\frac{1}{\sigma}}} \end{pmatrix}.$$

Definition 5 [43]: Let $\mathfrak{C} = \{c_1, c_2, \dots, c_n\}$ be an assembly of n attributes. Let there exist a priority relation between attributes $c_1 \succ c_2 \succ \cdots \succ c_n$, which signifies attribute c_i is superior to c_j if $i < j$. The value $c_i(t)$ indicates the performance of any alternative t regarding the attribute c_i such that $c_i(t) \in [0,1]$. The formula described below is known as the PAO.

$$PAO\left(c_i(t)\right) = \oplus_{i=1}^{n} \mathbb{P}_i c_i(t), \tag{7}$$

where $\mathbb{P}_i = \frac{w_i}{\sum_{i=1}^{n} w_i}$ with $\mathbb{P}_1 = 1$ and $\mathbb{P}_i = \prod_{\kappa=1}^{i-1} c_\kappa(t)\,; i = 1, 2, \dots, n$.

3. Schweizer-Sklar Weighted Prioritized Aggregation Operators for Spherical Fuzzy Information

This segment focuses on deriving the class of PAOs using SS-TN and SS-TCN within the framework of SFSs, namely the SFSSWPA and SFSSWPG operators. Also, we derived several desirable features of these operators.

Definition 6: Let $\mathcal{S}_j = \left(\mu_j, \nu_j, \eta_j\right); j = 2,3, \dots, n$, be a set of SFVs. Then the SFSSWPA operator is specified as:

$$SFSSWPA(\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_n) = w_1\mathcal{S}_1 \oplus w_2\mathcal{S}_2 \oplus \dots \oplus w_n\mathcal{S}_n = \oplus_{j=1}^{n} w_j\mathcal{S}_j, \tag{8}$$

where $w_j = \frac{\varpi_j w_j}{\sum_{j=1}^{n} \varpi_j w_j}$, $w_1 = 1$ and $w_j = \oplus_{\kappa}^{j-1} \mathfrak{S}(\mathcal{S}_\kappa), \kappa = 2,3, \dots, n$ Moreover, ϖ_j is the weight of jth SFV such that $\varpi_j \in [0, 1]$ and $\sum_{j=1}^{n} \varpi_j = 1$. Here, $\mathfrak{S}(\mathcal{S}_\kappa)$ expresses the score value of SFVs.

Theorem 1: Suppose that $\mathcal{S}_j = \left(\mu_j, \nu_j, \eta_j\right); j = 2,3, \dots, n$, be a set of SFVs and $\sigma < 0$. Then the aggregated outcome of the SFSSWPA operator is still an SFV and is described as:

$$SFSSWPA\left(\mu_j, \nu_j, \eta_j\right) = \left(\begin{array}{c} \sqrt{1 - \left(\sum_{j=1}^{n} w_j\left(1 - \mu_j^2\right)^\sigma - \sum_{j=1}^{n} w_j + 1\right)^{\frac{1}{\sigma}}} \\ \sqrt{\left(\sum_{j=1}^{n} w_j\left(\nu_j^2\right)^\sigma - \sum_{j=1}^{n} w_j + 1\right)^{\frac{1}{\sigma}}}, \\ \sqrt{\left(\sum_{j=1}^{n} w_j\left(\eta_j^2\right)^\sigma - \sum_{j=1}^{n} w_j + 1\right)^{\frac{1}{\sigma}}} \end{array}\right). \tag{9}$$

Proof: The desired result can be proved by using the principle of mathematical induction as follows:

Step 1: When n = 2, it follows that

$$w_1\mathcal{S}_1 = \left(\begin{array}{c} \sqrt{1 - \left(w_1(1 - \mu_1^2)^\sigma - (w_1 - 1)\right)^{\frac{1}{\sigma}}}, \\ \sqrt{\left(w_1(\nu_1^2)^\sigma - (w_1 - 1)\right)^{\frac{1}{\sigma}}}, \\ \sqrt{\left(w_1(\eta_1^2)^\sigma - (w_1 - 1)\right)^{\frac{1}{\sigma}}} \end{array}\right) \text{ and } w_2\mathcal{S}_2 = \left(\begin{array}{c} \sqrt{1 - \left(w_2(1 - \mu_2^2)^\sigma - (w_2 - 1)\right)^{\frac{1}{\sigma}}}, \\ \sqrt{\left(w_2(\nu_2^2)^\sigma - (w_2 - 1)\right)^{\frac{1}{\sigma}}}, \\ \sqrt{\left(w_2(\eta_2^2)^\sigma - (w_2 - 1)\right)^{\frac{1}{\sigma}}} \end{array}\right).$$

Thus, according to Eq. (9), it follows that

$SFSSWPA(\mathcal{S}_1, \mathcal{S}_2) = w_1\mathcal{S}_1 \oplus w_2\mathcal{S}_2$

$$= \begin{pmatrix} \sqrt{1-\left(w_1(1-\mu_1^2)^\sigma-(w_1-1)\right)^{\frac{1}{\sigma}}}, \\ \sqrt{\left(w_1(\nu_1^2)^\sigma-(w_1-1)\right)^{\frac{1}{\sigma}}}, \\ \sqrt{\left(w_1(\eta_1^2)^\sigma-(w_1-1)\right)^{\frac{1}{\sigma}}} \end{pmatrix} \oplus \begin{pmatrix} \sqrt{1-\left(w_2(1-\mu_2^2)^\sigma-(w_2-1)\right)^{\frac{1}{\sigma}}}, \\ \sqrt{\left(w_2(\nu_2^2)^\varphi-(w_2-1)\right)^{\frac{1}{\sigma}}}, \\ \sqrt{\left(w_2(\eta_2^2)^\varphi-(w_2-1)\right)^{\frac{1}{\sigma}}} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{\left(1-\left(\left(1-\left(1-\left(w_1(1-\mu_1^2)^\sigma-(w_1-1)\right)^{\frac{1}{\sigma}}\right)\right)^\sigma+\left(1-\left(1-\left(w_2(1-\mu_2^2)^\sigma-(w_2-1)\right)^{\frac{1}{\sigma}}\right)\right)^\sigma\right)-1\right)^{\frac{1}{\sigma}}}, \\ \sqrt{\left(\left(\left(w_1(\nu_1^2)^\sigma-(w_1-1)\right)^{\frac{1}{\sigma}}\right)^\sigma+\left(\left(w_2((\nu_1^2)^2)^\sigma-(w_2-1)\right)^{\frac{1}{\sigma}}\right)^\sigma-1\right)^{\frac{1}{\sigma}}}, \\ \sqrt{\left(\left(\left(w_1(\eta_1^2)^\sigma-(w_1-1)\right)^{\frac{1}{\sigma}}\right)^\sigma+\left(\left(w_2(\eta_2^2)^\sigma-(w_2-1)\right)^{\frac{1}{\sigma}}\right)^\sigma-1\right)^{\frac{1}{\sigma}}} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{\left(1-\left(\left(1-1+\left(w_1(1-\mu_1^2)^\sigma-(w_1-1)\right)^{\frac{1}{\sigma}}\right)^\sigma+\left(1-1+\left(w_2(1-\mu_2^2)^\sigma-(w_2-1)\right)^{\frac{1}{\sigma}}\right)^\sigma\right)-1\right)^{\frac{1}{\sigma}}}, \\ \sqrt{\left(\left(w_1(\nu_1^2)^\sigma-(w_1-1)\right)+\left(w_2(\nu_2^2)^\sigma-(w_2-1)\right)-1\right)^{\frac{1}{\sigma}}}, \\ \sqrt{\left(\left(w_1(\eta_1^2)^\sigma-(w_1-1)\right)+\left(w_2(\eta_2^2)^\sigma-(w_2-1)\right)-1\right)^{\frac{1}{\sigma}}} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{\left(1-\left(\left(\left(w_1(1-\mu_1^2)^\sigma-(w_1-1)\right)^{\frac{1}{\sigma}}\right)^\sigma+\left(\left(w_2(1-\mu_2^2)^\sigma-(w_2-1)\right)^{\frac{1}{\sigma}}\right)^\sigma\right)-1\right)^{\frac{1}{\sigma}}}, \\ \sqrt{(w_1(\nu_1^2)^\sigma-w_1+1+w_2(\nu_2^2)^\sigma-w_2+1-1)^{\frac{1}{\sigma}}}, \\ \sqrt{(w_1(\eta_1^2)^\sigma-w_1+1+w_2(\eta_2^2)^\sigma-w_2+1-1)^{\frac{1}{\sigma}}} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{\left(1-\left(\left(w_1(1-\mu_1^2)^\sigma-(w_1-1)\right)+\left(w_2(1-\mu_2^2)^\sigma-(w_2-1)\right)\right)-1\right)^{\frac{1}{\sigma}}}, \\ \sqrt{(w_1(\nu_1^2)^\sigma-w_1+1+w_2(\nu_2^2)^\sigma-w_2+1-1)^{\frac{1}{\sigma}}}, \\ \sqrt{(w_1(\eta_1^2)^\sigma-w_1+1+w_2(\eta_2^2)^\sigma-w_2+1-1)^{\frac{1}{\sigma}}} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{\left(1-\left(w_1(1-\mu_1^2)^\sigma+w_2(1-\mu_2^2)^\sigma-(w_1-1)-(w_2-1)\right)-1\right)^{\frac{1}{\sigma}}}, \\ \sqrt{(w_1(\nu_1^2)^\sigma+w_2(\nu_2^2)^\sigma-w_1-w_2+1)^{\frac{1}{\sigma}}}, \\ \sqrt{(w_1(\eta_1^2)^\sigma+w_2(\eta_2^2)^\sigma-w_1-w_2+1)^{\frac{1}{\sigma}}} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{(1-(w_1(1-\mu_1^2)^\sigma + w_2(1-\mu_2^2)^\sigma - w_1 + 1 - w_2 + 1) - 1)^{\frac{1}{\sigma}}}, \\ \sqrt{(w_1(\nu_1^2)^\sigma + w_2(\nu_1^2)^\sigma - w_1 - w_2 + 1)^{\frac{1}{\sigma}}}, \\ \sqrt{(w_1(\eta_1^2)^\sigma + w_2(\eta_2^2)^\sigma - w_1 - w_2 + 1)^{\frac{1}{\sigma}}} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{\left(1-(w_1(1-\mu_1^2)^\sigma + w_2(1-\mu_2^2)^\sigma - w_1 - w_2 + 1)\right)^{\frac{1}{\sigma}}}, \\ \sqrt{(w_1(\nu_1^2)^\sigma + w_2(\nu_1^2)^\sigma - w_1 - w_2 + 1)^{\frac{1}{\sigma}}}, \\ \sqrt{(w_1(\eta_1^2)^\sigma + w_2(\eta_2^2)^\sigma - w_1 - w_2 + 1)^{\frac{1}{\sigma}}} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{1-\left(\sum_{j=1}^{2} w_j\left(1-\mu_j^2\right)^\sigma - \sum_{j=1}^{2} w_j + 1\right)^{\frac{1}{\sigma}}} \\ \sqrt{\left(\sum_{j=1}^{2} w_j\left(\nu_j^2\right)^\sigma - \sum_{j=1}^{2} w_j + 1\right)^{\frac{1}{\sigma}}}, \\ \sqrt{\left(\sum_{j=1}^{2} w_j\left(\eta_j^2\right)^\sigma - \sum_{j=1}^{2} w_j + 1\right)^{\frac{1}{\sigma}}} \end{pmatrix}.$$

Therefore, for $n = 2$, the result is valid.

Step 2: We assume that Eq. (9) is valid for $n = \kappa$, then

$$SFSSWPA(\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_\kappa) = \begin{pmatrix} \sqrt{1-\left(\sum_{j=1}^{\kappa} w_j\left(1-\mu_j^2\right)^\sigma - \sum_{j=1}^{\kappa} w_j + 1\right)^{\frac{1}{\sigma}}} \\ \sqrt{\left(\sum_{j=1}^{\kappa} w_j\left(\nu_j^2\right)^\sigma - \sum_{j=1}^{\kappa} w_j + 1\right)^{\frac{1}{\sigma}}}, \\ \sqrt{\left(\sum_{j=1}^{\kappa} w_j\left(\eta_j^2\right)^\sigma - \sum_{j=1}^{\kappa} w_j + 1\right)^{\frac{1}{\sigma}}} \end{pmatrix}.$$

Step 3: To prove that Eq. (9) is true for $n = \kappa + 1$, consider

$$SFSSPA(\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_\kappa, \mathcal{S}_{\kappa+1}) = w_1\mathcal{S}_1 \oplus w_2\mathcal{S}_2 \oplus \dots \oplus w_\kappa\mathcal{S}_\kappa \oplus w_{\kappa+1}\mathcal{S}_{\kappa+1}$$
$$= \oplus_{j=1}^{\kappa} w_j\mathcal{S}_j \oplus w_{\kappa+1}\mathcal{S}_{\kappa+1}$$

$$= \begin{pmatrix} \sqrt{1-\left(\sum_{j=1}^{\kappa} w_j\left(1-\mu_j^2\right)^{\sigma}-\sum_{j=1}^{K} w_j+1\right)^{\frac{1}{\sigma}}}, \\ \sqrt{\left(\sum_{j=1}^{K} w_j\left(\nu_j^2\right)^{\sigma}-\sum_{j=1}^{K} w_j+1\right)^{\frac{1}{\sigma}}}, \\ \sqrt{\left(\sum_{j=1}^{K} w_j\left(\eta_j^2\right)^{\sigma}-\sum_{j=1}^{K} w_j+1\right)^{\frac{1}{\sigma}}} \end{pmatrix} \oplus \begin{pmatrix} \sqrt{1-\left(w_{K+1}(1-\mu_{k+1}^2)^{\sigma}-(w_{K+1}-1)\right)^{\frac{1}{\sigma}}} \\ \sqrt{\left(w_{K+1}(\nu_{K+1}^2)^{\sigma}-(w_{K+1}-1)\right)^{\frac{1}{\sigma}}}, \\ \sqrt{\left(w_{K+1}(\eta_{K+1}^2)^{\sigma}-(w_{K+1}-1)\right)^{\frac{1}{\sigma}}} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{1-\left(\sum_{j=1}^{\kappa+1} w_j\left(1-\mu_j^2\right)^{\sigma}-\sum_{j=1}^{\kappa+1} w_j+1\right)^{\frac{1}{\sigma}}}, \\ \sqrt{\left(\sum_{j=1}^{\kappa+1} w_j\left(\nu_j^2\right)^{\sigma}-\sum_{j=1}^{\kappa+1} w_j+1\right)^{\frac{1}{\sigma}}}, \\ \sqrt{\left(\sum_{j=1}^{\kappa+1} w_j\left(\eta_j^2\right)^{\sigma}-\sum_{j=1}^{\kappa+1} w_j+1\right)^{\frac{1}{\sigma}}} \end{pmatrix}.$$

Thus, the result is true for $n = \kappa + 1$. Hence, according to steps 1, 2, and 3, it becomes evident that the result is true for all $n \in \mathbb{Z}^+$.

According to Theorem 1, we can easily prove that the $SFSSWPA$ operator possesses the following characteristics.

Proposition 1: Presume that $\mathcal{S}_j = (\mu_j, \nu_j, \eta_j); j = 1, 2, \dots, n$ be a group of SFVs. Then,

(1). (*Idempotency*) If $\mathcal{S} = \mathcal{S}_j \ \forall\ j = 1, 2, \dots, n$, then

$$SFSSWPA(\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_n) = \mathcal{S}. \tag{10}$$

(2). (*Monotonicity*) For any two families of SFVs $\mathcal{S}_j = (\mu_j, \nu_j, \eta_j)$ and $\widetilde{\mathcal{S}_j} = (\mu_j^*, \nu_j^*, \eta_j^*)$ such that $\mu_j \le \mu_j^*$, $\nu_j \ge \nu_j^*$ and $\eta_j \ge \eta_j^*$. Then,

$$SFSSWPA(\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_n) \le SFSSWPA(\widetilde{\mathcal{S}_1}, \widetilde{\mathcal{S}_2}, \dots, \widetilde{\mathcal{S}_n}). \tag{11}$$

(3). (*Boundedness*) Let $\mathcal{S}_j^- = \{min(\mu_j), \max(\nu_j), \max(\eta_j)\}$ and $\mathcal{S}_j^+ = \{max(\mu_j), \min(\nu_j), \min(\eta_j)\}$ be the smallest and largest SFVs, then

$$\mathcal{S}_j^- \le SFSSWPA(\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_n) \le \mathcal{S}_j^+. \tag{12}$$

Proof:

(1). Consider $\mathcal{S} = \mathcal{S}_j = (\mu_j, \nu_j, \eta_j)\ \forall\ j$. Then, by using Eq. (9), we have

$$SFSSWPA(\mathcal{S},\mathcal{S},\ldots,\mathcal{S}) = \begin{pmatrix} \sqrt{1-\left(\sum_{j=1}^{n} w_j(1-\mu^2)^\sigma - \sum_{j=1}^{n} w_j + 1\right)^{\frac{1}{\sigma}}}, \\ \sqrt{\left(\sum_{j=1}^{n} w_j(\nu^2)^\sigma - \sum_{j=1}^{n} w_j + 1\right)^{\frac{1}{\sigma}}}, \\ \sqrt{\left(\sum_{j=1}^{n} w_j(\eta^2)^\sigma - \sum_{j=1}^{n} w_j + 1\right)^{\frac{1}{\sigma}}} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{1-((1-\mu^2)^\sigma - 1 + 1)^{\frac{1}{\sigma}}}, \\ \sqrt{((\nu^2)^\sigma - 1 + 1)^{\frac{1}{\sigma}}}, \\ \sqrt{((\eta^2)^\sigma - 1 + 1)^{\frac{1}{\sigma}}}, \end{pmatrix}, \sum_{j=1}^{n} w_j = 1$$

$$= \begin{pmatrix} \sqrt{1-((1-\mu^2)^\sigma)^{\frac{1}{\sigma}}}, \\ \sqrt{((\nu^2)^\sigma)^{\frac{1}{\sigma}}}, \\ \sqrt{((\eta^2)^\sigma)^{\frac{1}{\sigma}}}, \end{pmatrix} = \begin{pmatrix} \sqrt{1-(1-\mu^2)}, \\ \sqrt{(\nu^2)}, \\ \sqrt{(\eta^2)}, \end{pmatrix} = (\mu,\nu,\eta) = \mathcal{S}.$$

(2). Assume that $\mathcal{S}_j \ \widetilde{\leq \mathcal{S}_j}$, then $\mu_j \leq \mu_j^*$, $\nu_j \geq \nu_j^*$ and $\eta_j \geq \eta_j^*$. Now consider

$$\mu_j \leq \mu_j^* \Rightarrow 1-\mu_j \leq 1-\mu_j^*$$
$$\Rightarrow \left(1-\mu_j\right)^\varphi \geq \left(1-\mu_j^*\right)^\sigma$$
$$\Rightarrow \sum_{j=1}^{n} w_j \left(1-\mu_j\right)^\sigma \geq \sum_{j=1}^{n} w_j \left(1-\mu_j^*\right)^\sigma$$
$$\Rightarrow \sum_{j=1}^{n} w_j \left(1-\mu_j\right)^\sigma - \sum_{j=1}^{n} w_j + 1 \geq \sum_{j=1}^{n} w_j \left(1-\mu_j^*\right)^\sigma - \sum_{j=1}^{n} w_j + 1$$
$$\Rightarrow \left(\sum_{j=1}^{n} w_j \left(1-\mu_j\right)^\sigma - \sum_{j=1}^{n} w_j + 1\right)^{\frac{1}{\sigma}} \geq \left(\sum_{j=1}^{n} w_j \left(1-\mu_j^*\right)^\sigma - \sum_{j=1}^{n} w_j + 1\right)^{\frac{1}{\sigma}}$$
$$\Rightarrow \left(\sum_{j=1}^{n} w_j \left(1-\mu_j\right)^\sigma - \sum_{j=1}^{n} w_j + 1\right)^{\frac{1}{\sigma}} \leq \left(\sum_{j=1}^{n} w_j \left(1-\mu_j^*\right)^\sigma - \sum_{j=1}^{n} w_j + 1\right)^{\frac{1}{\sigma}}. \quad (13)$$

Similarly,

$$\nu_j \geq \nu_j^* \Rightarrow {\nu_j}^\sigma \geq {\nu_j^*}^\sigma$$

$$\Rightarrow \sum_{j=1}^{n} w_j {\nu_j}^\sigma \geq \sum_{j=1}^{n} w_j {\nu_j^*}^\sigma$$
$$\Rightarrow \sum_{j=1}^{n} w_j {\nu_j}^\sigma - \sum_{j=1}^{n} w_j + 1 \geq \sum_{j=1}^{n} w_j {\nu_j^*}^\sigma - \sum_{j=1}^{n} w_j + 1$$
$$\Rightarrow \left(\sum_{j=1}^{n} w_j {\nu_j}^\sigma - \sum_{j=1}^{n} w_j + 1\right)^{\frac{1}{\sigma}} \geq \left(\sum_{j=1}^{n} w_j {\nu_j^*}^\sigma - \sum_{j=1}^{n} w_j + 1\right)^{\frac{1}{\sigma}}. \quad (14)$$

Further,

$$\eta_j \geq \eta_j^* \Rightarrow {\eta_j}^\sigma \geq {\eta_j^*}^\sigma$$

$$\Rightarrow \sum_{j=1}^{n} w_j {\eta_j}^{\sigma} \geq \sum_{j=1}^{n} w_j {\eta_j^{*}}^{\sigma}$$
$$\Rightarrow \sum_{j=1}^{n} w_j {\eta_j}^{\sigma} - \sum_{j=1}^{n} w_j + 1 \geq \sum_{j=1}^{n} w_j {\eta_j^{*}}^{\sigma} - \sum_{j=1}^{n} w_j + 1$$
$$\Rightarrow \left(\sum_{j=1}^{n} w_j {\eta_j}^{\sigma} - \sum_{j=1}^{n} w_j + 1\right)^{\frac{1}{\sigma}} \geq \left(\sum_{j=1}^{n} w_j {\eta_j^{*}}^{\sigma} - \sum_{j=1}^{n} w_j + 1\right)^{\frac{1}{\sigma}}. \tag{15}$$

Merging inequalities (13) to (15), we get

$$SFSSWPA(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n) \leq SFSSWPA(\widetilde{\mathcal{S}_1}, \widetilde{\mathcal{S}_2}, \ldots, \widetilde{\mathcal{S}_n}).$$

(3). Since

$$SFSSWPA(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n) \geq SFSSWPA(\mathcal{S}_1^-, \mathcal{S}_2^-, \ldots, \mathcal{S}_n^-) = \mathcal{S}_j^-$$

and

$$SFSSWPA(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n) \leq SFSSWPA(\mathcal{S}_1^+, \mathcal{S}_2^+, \ldots, \mathcal{S}_n^+) = \mathcal{S}_j^+.$$

Therefore, $\mathcal{S}_j^- \leq SFSSWPA(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n) \leq \mathcal{S}_j^+$.

Special case: When $\nu_j = 0$; for all $j = 1, 2, \ldots, n$, then the SFSSWPA operator will be reduced to the PFSSPWA operator proposed by Yang and Zhang [30].

Definition 7: Suppose that $\mathcal{S}_j = (\mu_j, \nu_j, \eta_j); j = 1, 2, \ldots, n$ be a group of SFVs and $W_j = (\varpi_1, \varpi_2, \ldots, \varpi_n)$ be the weight vector such that $\varpi_j \in [0, 1]$ with $\sum_{j=1}^{n} \varpi_j = 1$. Then, the notion of the SFSSWPG operator is postulated as follows:

$$SFSSWPG(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n) = \mathcal{S}_1^{w_1} \otimes \mathcal{S}_2^{w_2} \otimes \ldots \otimes \mathcal{S}_n^{w_n} = \otimes_{j=1}^{n} \mathcal{S}_j^{w_j}, \tag{16}$$

where, $w_j = \frac{\varpi_j w_j}{\sum_{j=1}^{n} \varpi_j w_j}$, $w_1 = 1$ and $w_j = \bigoplus_{\kappa}^{j-1} \mathfrak{S}(\mathcal{S}_\kappa), \kappa = 2, 3, \ldots, n$.

Theorem 2: Let $\mathcal{S}_j = (\mu_j, \nu_j, \eta_j); j = 1, 2, \ldots, n$ be a collection of SFVs. Then, the fused value of the SFSSWPA operator is still an SFV, that is:

$$SFSSWPG(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n) = \begin{pmatrix} \sqrt{\left(\sum_{j=1}^{n} w_j (\nu_j^2)^{\sigma} - \sum_{j=1}^{n} w_j + 1\right)^{\frac{1}{\sigma}}}, \\ \sqrt{1 - \left(\sum_{j=1}^{n} w_j (1 - \eta_j^2)^{\sigma} - \sum_{j=1}^{n} w_j + 1\right)^{\frac{1}{\sigma}}} \\ \sqrt{1 - \left(\sum_{j=1}^{n} w_j (1 - \mu_j^2)^{\sigma} - \sum_{j=1}^{n} w_j + 1\right)^{\frac{1}{\sigma}}} \end{pmatrix}. \tag{17}$$

Proof: Analogous to the proof of Theorem 1.

Like those of the SFSSWPA operators, the SFSSWPG operator also obeys the properties of idempotency, monotonicity, and boundedness, which are specified as follows.

Proposition 2: Let $\mathcal{S}_j = (\mu_j, \nu_j, \eta_j); j = 1, 2, \ldots, n$ be an assemblage of SFVs. Then, the SFSSWPG operator owns the following properties:

(1). (*Idempotency*) When $\mathcal{S} = \mathcal{S}_j \ \forall\, j = 1, 2, \ldots, n$, then

$$SFSSWPG(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n) = \mathcal{S}. \tag{18}$$

(2). (*Monotonicity*) For any two collections of SFVs $\mathcal{S}_j = (\mu_j, \nu_j, \eta_j)$ and $\widetilde{\mathcal{S}_j} = (\mu_j^*, \nu_j^*, \eta_j^*)$ such that $\mu_j \leq \mu_j^*$, $\nu_j \geq \nu_j^*$ and $\eta_j \geq \eta_j^*$. Then,

$$SFSSWPG(\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_n) \leq SFSSWPG(\widetilde{\mathcal{S}_1}, \widetilde{\mathcal{S}_2}, \dots, \widetilde{\mathcal{S}_n}). \tag{19}$$

(3). (*Boundedness*) Let $\mathcal{S}_j^- = \{min(\mu_j), \max(\nu_j), \max(\eta_j)\}$ and $\mathcal{S}_j^+ = \{max(\mu_j), \min(\nu_j), \min(\eta_j)\}$ be the smallest and largest SFVs, then

$$\mathcal{S}_j^- \leq SFSSWPG(\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_n) \leq \mathcal{S}_j^+. \tag{20}$$

Special case: If $\nu_j = 0$; for all $j = 1, 2, \dots, n$, then the SFSSWPG operator is transformed to the PFSSPWG operator anticipated by Yang and Zhang [30].

4. MADM Approach with Spherical Fuzzy Information

This section employs the suggested SF Schweizer-Sklar PAOs to formulate a novel MADM framework in SF contexts. The following assumptions are used to elucidate the problem under consideration.

4.1. Problem Description

Let $\mathcal{A} = \{\aleph_1, \aleph_2, \dots, \aleph_m\}$ signifies a collection of alternative and consider a gathering of attributes $\mathfrak{C} = \{e_1, e_2, \dots, e_n\}$ that fulfills the prioritization condition: $e_1 \succ e_2 \succ \cdots \succ e_n$, which reveals that the attribute $e_1$has higher e_j, $\forall i < j$. Assume that $\mathbb{W} = (\varpi_1, \varpi_2, \dots, \varpi_n)^T$ be the weight vector of the attributes with constraint $\varpi_j \in [0, 1]$ with $\sum_{j=1}^{n} \varpi_j$. Presume that $\mathbb{M} = [\mathcal{S}_{ij}]_{m\times n} = [(\mu_{ij}, \nu_{ij}, \eta_{ij})]_{m\times n}$ is the SF decision matrix provided by an expert to evaluate the alternative $\aleph_i (i = 1,2, \dots, m)$ regarding the attribute $e_j (j = 1,2, \dots, n)$ while adhering to the condition $0 \leq \mu_{ij}^2 + \nu_{ij}^2 + \eta_{ij}^2 \leq 1$.
Considering these factors, the step-by-step process of the projected MADM approach is systematically outlined below.
Step 1: Building upon the preceding analysis, the expert evaluations for each alternative, concerning their corresponding attributes, are systematically arranged in the form of the SF decision matrix described:

$$\mathbb{M} = [\mathcal{S}_{ij}]_{m\times n} = [(\mu_{ij}, \nu_{ij}, \eta_{ij})]_{m\times n} \tag{21}$$

Step 2: In the framework of MCDM paradigms, attributes are generally characterized into two types: benefit-type and cost-type. To mitigate the impact of differing attribute types, it is vital to standardize all attributes by converting them into a uniform type. Determine the normalized matrix, denoted as $\widetilde{\mathbb{N}} = [\widetilde{\mathcal{S}_{\iota\jmath}}]_{m\times n}$, by using the following formulation:

$$\widetilde{\mathcal{S}_{\iota\jmath}} = \begin{cases} \mathcal{S}_{ij} = (\mu_{ij}, \nu_{ij}, \eta_{ij}); & \text{for benifit type attribute} \\ \mathcal{S}_{ij}^c = (\eta_{ij}, \nu_{ij}, \mu_{ij}); & \text{for cost type attribute,} \end{cases} \tag{22}$$

where $\mathcal{S}_{ij}^c$ signifies the complement of $\mathcal{S}_{ij}$.
Step 3: Determine the prioritized weighted values $w_j = \frac{w_j}{\sum_{j=1}^{n} w_j}$, where $w_1 = 1$ and $w_j = \bigoplus_{\kappa=1}^{j-1} \mathfrak{S}(\mathcal{S}_\kappa), \kappa = 2,3, \dots, n$.
Step 4: Aggregate the SFVs of $\mathcal{S}_{ij}$ associated to alternatives $\aleph_i (i = 1,2, \dots, m)$ under attributes $e_j (j = 1,2, \dots, n)$ by using SFSSWPA and SFSSWPG operators.
Step 5: Evaluate the scores the aggregated SFVs acquired in Step 4.

Step 6: The ranking of alternatives $\aleph_i$ $(i = 1,2, \dots, m)$ is determined based on their respective score values. The greater the score value, the better the alternative is.

5. Numerical Illustration

This segment offers an application of the MADM problem employing SF information to validate the practical implementation of the planned methodology.
The transition toward sustainable energy systems is one of the most pressing priorities in contemporary environmental policy and industrial planning. Renewable energy sources reduce dependence on fossil fuels, mitigate greenhouse gas emissions, and support long-term ecological balance while meeting growing global energy demand. Selecting the most suitable renewable energy source for a given region or organization, however, is a genuinely MADM: environmental benefit must be weighed against economic feasibility, the reliability and availability of the underlying resource, and the maturity of the technology required to harness it. These attributes are frequently uncertain, interdependent, and unevenly important to decision-makers, which makes the problem a natural candidate for SF prioritized aggregation. Selecting an appropriate renewable energy source further requires accounting for regional resource endowments, infrastructure readiness, and policy incentives, all of which experts must judge under considerable vagueness and hesitancy, precisely the setting SFSs are designed to capture.

5.1. Key Alternatives

Environmental impact reduction reflects the extent to which an energy source lowers carbon emissions, land degradation, and pollution relative to conventional fossil-fuel alternatives. Economic viability captures installation costs, operating expenses, and return on investment relative to expected energy output. Resource availability and reliability describe how consistently and abundantly the underlying energy resource (sunlight, wind, water flow, biomass feedstock, or geothermal heat) can be accessed within a given region. Technological maturity and scalability reflect how well-established the harvesting and grid-integration technology is, and how readily capacity can be expanded to meet rising demand.
Now, we consider five different types of renewable energy sources: Solar Energy ($\aleph_1$), Wind Energy ($\aleph_2$), Hydropower ($\aleph_3$), Biomass Energy ($\aleph_4$), and Geothermal Energy ($\aleph_5$).

5.2. Key Attributes

The decision to assess these renewable energy sources is based on four different types of attributes, whose details are as follows.

- Environmental Impact (e_1): This attribute captures the degree to which an energy source reduces greenhouse gas emissions, air and water pollution, and ecological disruption compared with conventional generation. Sources with minimal emissions and a small land- or water-use footprint score more favorably on this criterion.
- Economic Viability (e_2): Economic viability reflects the cost-effectiveness of an energy source across its lifecycle, including capital investment, maintenance costs, and the levelized cost of energy. Sources that achieve competitive returns with lower subsidy dependence are considered more advantageous.

- Resource Availability and Reliability (e_3): This attribute measures the consistency and abundance of the underlying resource within a given region, including its resilience to seasonal or climatic variation. Energy sources with stable, predictable output are preferred, as intermittency raises the cost of grid balancing and storage.
- Technological Maturity and Scalability (e_4): Technological maturity assesses how well-developed the harvesting and integration technology is, along with the ease of scaling capacity to meet growing demand. Mature, modular technologies that integrate readily with existing grid infrastructure score more favorably.

5.3. Decision-Making Process

The empirical validation using real standard internet router specifications demonstrates the clear superiority of our proposed SFSSWPA and SFSSWPG operators. Subsequent comparative analysis against extant MADM models reveals that our approach yields highly stable rankings, even under fluctuating parameter settings. Importantly, security and latency criteria consistently emerge as primary drivers, overturning cost-centric procurement heuristics. This empirical evidence provides municipal chief technology officers with quantifiable assurance, enabling evidence-based infrastructure investments that enhance city-wide network resilience and operational continuity.

When choosing a renewable energy source, considering these attributes can help meet specific requirements, such as emissions targets, budget constraints, and regional resource conditions. The choice may depend on geographic location, grid infrastructure, and policy priorities. Let $\mathbb{W} = (0.30, 0.35, 0.15, 0.20)^T$ be the associated weight vector, which represents the relative importance of each attribute. The decision-expert evaluates the five alternatives $\aleph_i (i = 1,2, \ldots ,5)$ using SFVs based on the attribute $e_j (j = 1,2, \ldots ,4)$. The stepwise computational process of the devised approach for the considered MADM problem is outlined below:

Step 1: We organize the information provided by a decision expert using SFVs, which is displayed in Table 1.

Table 1
SF decision matrix

	e_1	e_2	e_3	e_4
$\aleph_1$	$(0.4, 0.3, 0.2)$	$(0.5, 0.3, 0.1)$	$(0.2, 0.3, 0.2)$	$(0.4, 0.1, 0.2)$
$\aleph_2$	$(0.5, 0.2, 0.1)$	$(0.1, 0.2, 0.2)$	$(0.6, 0.1, 0.2)$	$(0.3, 0.2, 0.1)$
$\aleph_3$	$(0.6, 0.5, 0.2)$	$(0.1, 0.3, 0.3)$	$(0.6, 0.1, 0.1)$	$(0.2, 0.3, 0.1)$
$\aleph_4$	$(0.4, 0.1, 0.3)$	$(0.4, 0.3, 0.1)$	$(0.3, 0.3, 0.2)$	$(0.2, 0.1, 0.1)$
$\aleph_5$	$(0.3, 0.4, 0.1)$	$(0.3, 0.4, 0.1)$	$(0.3, 0.2, 0.1)$	$(0.4, 0.1, 0.2)$

Step 2: Since there is no cost-type attribute, there is no need for normalization.

Step 3: We calculate the prioritized weighted values are portrayed in Table 2.

Step 4: On the ground of SFSSWPA and SFSSWPG operators, the aggregated SFVs associated to alternatives $\aleph_i (i = 1,2, \ldots ,5)$ concerning four attributes (by fixing $\sigma = -1$) are determined in Table 3.

Table 2
Weighted prioritized values

Alternative	Score values	$w_\tau = \frac{w_\tau}{\sum_{\tau=1}^{n} w_\tau}$	$\varpi_\tau e_\tau$	$\sum_{i=1}^{m} \varpi_\tau e_\tau$	e_τ
$\aleph_1$	$\varpi_1 e_1 = 0.6767$, $\varpi_2 e_2 = 0.7167$, $\varpi_3 e_3 = 0.6367$, $\varpi_4 e_4 = 0.7033$	$e_1 = 1$, $e_2 = 0.6767$, $e_3 = 0.4849$, $e_4 = 0.3087$	$\varpi_1 e_1 = 0.3000$, $\varpi_2 e_2 = 0.2368$, $\varpi_3 e_3 = 0.0727$, $\varpi_4 e_4 = 0.0617$	0.6713	$e_1 = 0.4469$, $e_2 = 0.3528$, $e_3 = 0.1084$, $e_4 = 0.0920$
$\aleph_2$	$\varpi_1 e_1 = 0.7333$, $\varpi_2 e_2 = 0.6433$, $\varpi_3 e_3 = 0.7700$, $\varpi_4 e_4 = 0.6800$	$e_1 = 1$, $e_2 = 0.7333$, $e_3 = 0.4718$, $e_4 = 0.3633$	$\varpi_1 e_1 = 0.3000$, $\varpi_2 e_2 = 0.2567$, $\varpi_3 e_3 = 0.0708$, $\varpi_4 e_4 = 0.0727$	0.7001	$e_1 = 0.4285$, $e_2 = 0.3666$, $e_3 = 0.1011$, $e_4 = 0.1038$
$\aleph_3$	$\varpi_1 e_1 = 0.6900$, $\varpi_2 e_2 = 0.6100$, $\varpi_3 e_3 = 0.7800$, $\varpi_4 e_4 = 0.6467$	$e_1 = 1$, $e_2 = 0.6900$, $e_3 = 0.4209$, $e_4 = 0.3283$	$\varpi_1 e_1 = 0.3000$, $\varpi_2 e_2 = 0.2415$, $\varpi_3 e_3 = 0.0631$, $\varpi_4 e_4 = 0.0657$	0.6703	$e_1 = 0.4476$, $e_2 = 0.3603$, $e_3 = 0.0942$, $e_4 = 0.0980$
$\aleph_4$	$\varpi_1 e_1 = 0.6867$, $\varpi_2 e_2 = 0.6867$, $\varpi_3 e_3 = 0.6533$, $\varpi_4 e_4 = 0.6733$	$e_1 = 1$, $e_2 = 0.6867$, $e_3 = 0.4715$, $e_4 = 0.3081$	$\varpi_1 e_1 = 0.3000$, $\varpi_2 e_2 = 0.2403$, $\varpi_3 e_3 = 0.0707$, $\varpi_4 e_4 = 0.0616$	0.6727	$e_1 = 0.4460$, $e_2 = 0.3573$, $e_3 = 0.1051$, $e_4 = 0.0916$
$\aleph_5$	$\varpi_1 e_1 = 0.6400$, $\varpi_2 e_2 = 0.6400$, $\varpi_3 e_3 = 0.6800$, $\varpi_4 e_4 = 0.7033$	$e_1 = 1$, $e_2 = 0.6400$, $e_3 = 0.4096$, $e_4 = 0.2785$	$\varpi_1 e_1 = 0.3000$, $\varpi_2 e_2 = 0.2240$, $\varpi_3 e_3 = 0.0614$, $\varpi_4 e_4 = 0.0557$	0.6411	$e_1 = 0.4679$, $e_2 = 0.3494$, $e_3 = 0.0958$, $e_4 = 0.0869$

Table 3
Aggregated values

	SFSSPWA operator	SFSSPWG operator
$\aleph_1$	$(0.4284, 0.2277, 0.1394)$	$(0.3654, 0.2884, 0.1721)$
$\aleph_2$	$(0.4196, 0.1752, 0.1241)$	$(0.1585, 0.1925, 0.1558)$
$\aleph_3$	$(0.4879, 0.2477, 0.1705)$	$(0.1581, 0.4044, 0.2307)$
$\aleph_4$	$(0.3785, 0.1303, 0.1381)$	$(0.3434, 0.2206, 0.2244)$
$\aleph_5$	$(0.3107, 0.2485, 0.1034)$	$(0.3059, 0.3719, 0.1126)$

Step 5: Based on the information in Table 3, the score values are computed in Table 4.

Table 4
Score values

	SFSSPWA operator	SFSSPWG operator
$\mathfrak{S}(\aleph_1)$	0.7041	0.6736
$\mathfrak{S}(\aleph_2)$	0.7100	0.6546
$\mathfrak{S}(\aleph_3)$	0.7159	0.6028
$\mathfrak{S}(\aleph_4)$	0.7024	0.6730
$\mathfrak{S}(\aleph_5)$	0.6747	0.6475

Step 6: Based on the score values, we obtain the ranking outcomes presented in Table 5.

Table 5
Ranking of alternatives

Operators	Ranking and ordering
$\boldsymbol{SFSSPWA}$	$\aleph_3 \succcurlyeq \aleph_2 \succcurlyeq \aleph_1 \succcurlyeq \aleph_4 \succcurlyeq \aleph_5$
$\boldsymbol{SFSSPWG}$	$\aleph_1 \succcurlyeq \aleph_4 \succcurlyeq \aleph_2 \succcurlyeq \aleph_5 \succcurlyeq \aleph_3$

Considering Table 5, we can notice that under the SFSSWPA and SFSSWPG operators, the optimal alternatives are hydropower ($\aleph_3$) and Solar Energy ($\aleph_1$), respectively. The variation in optimal candidates among the two operators is not incidental; it reflects their distinct aggregation mechanism. The SFSSWPA operator is built on the SS-TCN, which is compensatory: strong performance on high-priority attributes can offset a weak score elsewhere. Under this logic, Hydropower ($\aleph_3$) is favored, since its high scores on Routing/Environmental (e_1) and Security/Resource-related (e_3) attributes compensate for its comparatively weak e_2 value. In contrast, the SFSSWPG operator is built on the SS-TN, which is non-compensatory: the weakest attribute dominates the fused outcome regardless of strength elsewhere.

6. Comparison and Discussion

This section offers a comprehensive sensitivity analysis and a comparative assessment of the planned methodology against state-of-the-art approaches, highlighting its efficiency, superiority, and practical utility.

6.1 Sensitivity Analysis

The SS-TN and SS-TCN incorporate a regulating parameter, σ, which increases their flexibility and efficiency in handling fuzzy information. This adaptability enables improved handling of uncertainty, making these models particularly significant in applications requiring robust fuzzy data processing.

In this subsection, we conduct a comprehensive sensitivity analysis to estimate the influence of the critical parameter σ. Specifically, we scrutinize its behavior across a range of values of σ from -1 to -100 within the SFSSWPA and SFSSWPG operators. The resulting scores and ranking orders obtained through these operators are systematically presented in Tables 6 and 7, respectively. Additionally, the visual ranking result is shown in Figures 1 and 2. According to Tables 6 and 7, it is necessary to notice that the corresponding score values of the alternatives are monotonically increasing as the inputs of the parameter σ decrease. This reveals that the smaller value of the parameter σ must be allotted for making pessimistic decisions, whereas for making optimistic decisions, the expert chooses a greater value of the parameter σ.

Table 6
Results obtained by SFSSWPA with varying input of σ

Values of σ	$Score\ values$					**Ranking**
	$\aleph_1$	$\aleph_2$	$\aleph_3$	$\aleph_4$	$\aleph_5$	
-1	0.7041	0.7100	0.7159	0.7024	0.6747	$\aleph_3 \succcurlyeq \aleph_2 \succcurlyeq \aleph_1 \succcurlyeq \aleph_4 \succcurlyeq \aleph_5$
-5	0.7213	0.7284	0.7534	0.7076	0.6904	$\aleph_3 \succcurlyeq \aleph_2 \succcurlyeq \aleph_1 \succcurlyeq \aleph_4 \succcurlyeq \aleph_5$
-20	0.7313	0.7556	0.7727	0.7105	0.6940	$\aleph_3 \succcurlyeq \aleph_2 \succcurlyeq \aleph_1 \succcurlyeq \aleph_4 \succcurlyeq \aleph_5$
-35	0.7357	0.7654	0.7758	0.7115	0.6970	$\aleph_3 \succcurlyeq \aleph_2 \succcurlyeq \aleph_1 \succcurlyeq \aleph_4 \succcurlyeq \aleph_5$
-55	0.7384	0.7707	0.7774	0.7121	0.7011	$\aleph_3 \succcurlyeq \aleph_2 \succcurlyeq \aleph_1 \succcurlyeq \aleph_4 \succcurlyeq \aleph_5$
-70	0.7394	0.7728	0.7779	0.7124	0.7034	$\aleph_3 \succcurlyeq \aleph_2 \succcurlyeq \aleph_1 \succcurlyeq \aleph_4 \succcurlyeq \aleph_5$
-80	0.7399	0.7737	0.7782	0.7125	0.7046	$\aleph_3 \succcurlyeq \aleph_2 \succcurlyeq \aleph_1 \succcurlyeq \aleph_4 \succcurlyeq \aleph_5$
-90	0.7403	0.7744	0.7784	0.7126	0.7056	$\aleph_3 \succcurlyeq \aleph_2 \succcurlyeq \aleph_1 \succcurlyeq \aleph_4 \succcurlyeq \aleph_5$
-100	0.7406	0.7750	0.7786	0.7127	0.7063	$\aleph_3 \succcurlyeq \aleph_2 \succcurlyeq \aleph_1 \succcurlyeq \aleph_4 \succcurlyeq \aleph_5$

At the same time, considering Table 6 and Figure 1, the final ranking outcomes of the alternatives obtained by the SFSSWPA operator are consistent, with $\aleph_3$ as an optimal alternative, no matter how the inputs of the parameter σ are changed. Meanwhile, we can find from Table 7 and Figure 2 that the ranking of alternatives alters for $\sigma = -1, -5$ in the framework of the SFSSWPG operator, while for $-100 \leq \sigma \leq -20$, the ranking outcomes remain consistent, showing that $\aleph_2$ an optimal alternative.

Based on the preceding discussion, it is evident that the proposed approach demonstrates a certain degree of stability with respect to the parameter σ.

Table 7
Ranking outcomes obtained by SFSSWPG with different values of σ

Values of σ	***Score values***					**Ranking**
	$\aleph_1$	$\aleph_2$	$\aleph_3$	$\aleph_4$	$\aleph_5$	
-1	0.6736	0.6546	0.6028	0.6730	0.6475	$\aleph_1 \succcurlyeq \aleph_4 \succcurlyeq \aleph_2 \succcurlyeq \aleph_5 \succcurlyeq \aleph_3$
-5	0.6494	0.6501	0.5919	0.6529	0.6452	$\aleph_4 \succcurlyeq \aleph_2 \succcurlyeq \aleph_1 \succcurlyeq \aleph_5 \succcurlyeq \aleph_3$
-20	0.6423	0.6488	0.5760	0.6393	0.6416	$\aleph_2 \succcurlyeq \aleph_1 \succcurlyeq \aleph_5 \succcurlyeq \aleph_4 \succcurlyeq \aleph_3$
-35	0.6408	0.6480	0.5698	0.6335	0.6402	$\aleph_2 \succcurlyeq \aleph_1 \succcurlyeq \aleph_5 \succcurlyeq \aleph_4 \succcurlyeq \aleph_3$
-55	0.6397	0.6472	0.5657	0.6292	0.6391	$\aleph_2 \succcurlyeq \aleph_1 \succcurlyeq \aleph_4 \succcurlyeq \aleph_5 \succcurlyeq \aleph_3$
-70	0.6392	0.6467	0.5639	0.6273	0.6385	$\aleph_2 \succcurlyeq \aleph_1 \succcurlyeq \aleph_5 \succcurlyeq \aleph_4 \succcurlyeq \aleph_3$
-80	0.6390	0.6464	0.5631	0.6264	0.6380	$\aleph_2 \succcurlyeq \aleph_1 \succcurlyeq \aleph_5 \succcurlyeq \aleph_4 \succcurlyeq \aleph_3$
-90	0.6388	0.6462	0.5624	0.6257	0.6376	$\aleph_2 \succcurlyeq \aleph_1 \succcurlyeq \aleph_5 \succcurlyeq \aleph_4 \succcurlyeq \aleph_3$
-100	0.6300	0.6460	0.5618	0.6251	0.6372	$\aleph_2 \succcurlyeq \aleph_1 \succcurlyeq \aleph_5 \succcurlyeq \aleph_4 \succcurlyeq \aleph_3$

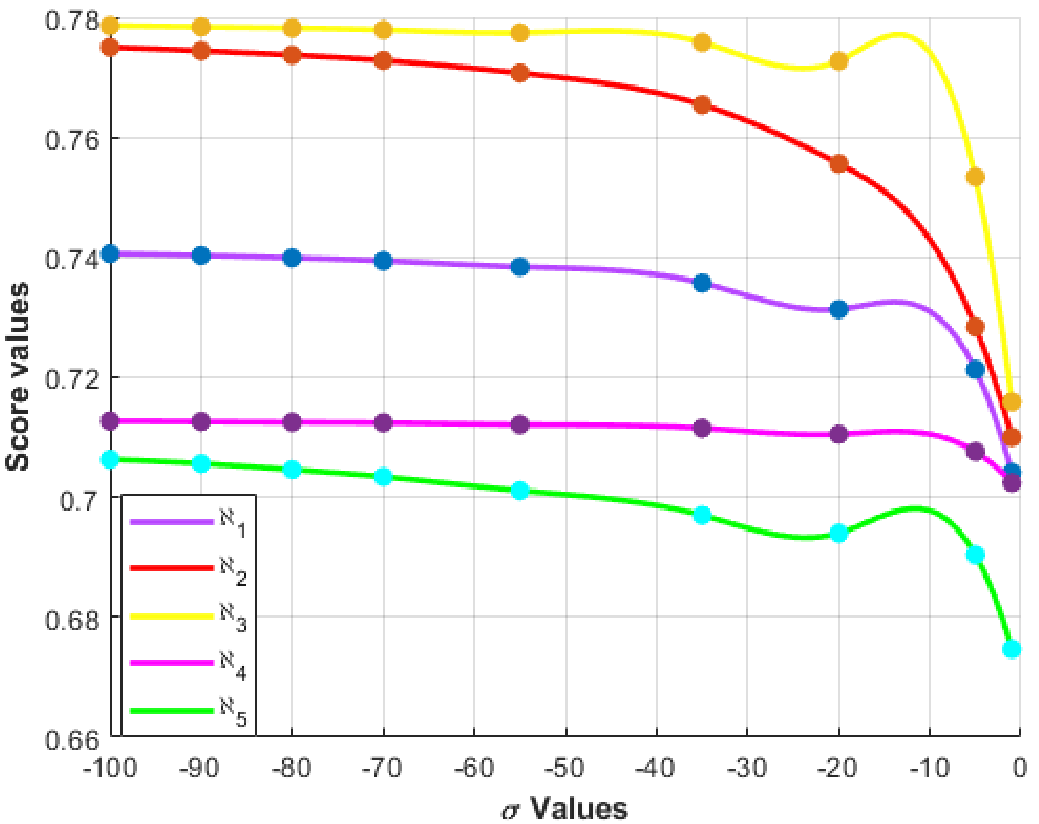

Fig. 1. Impact of $\boldsymbol{\sigma}$ under SFSSWPA operator

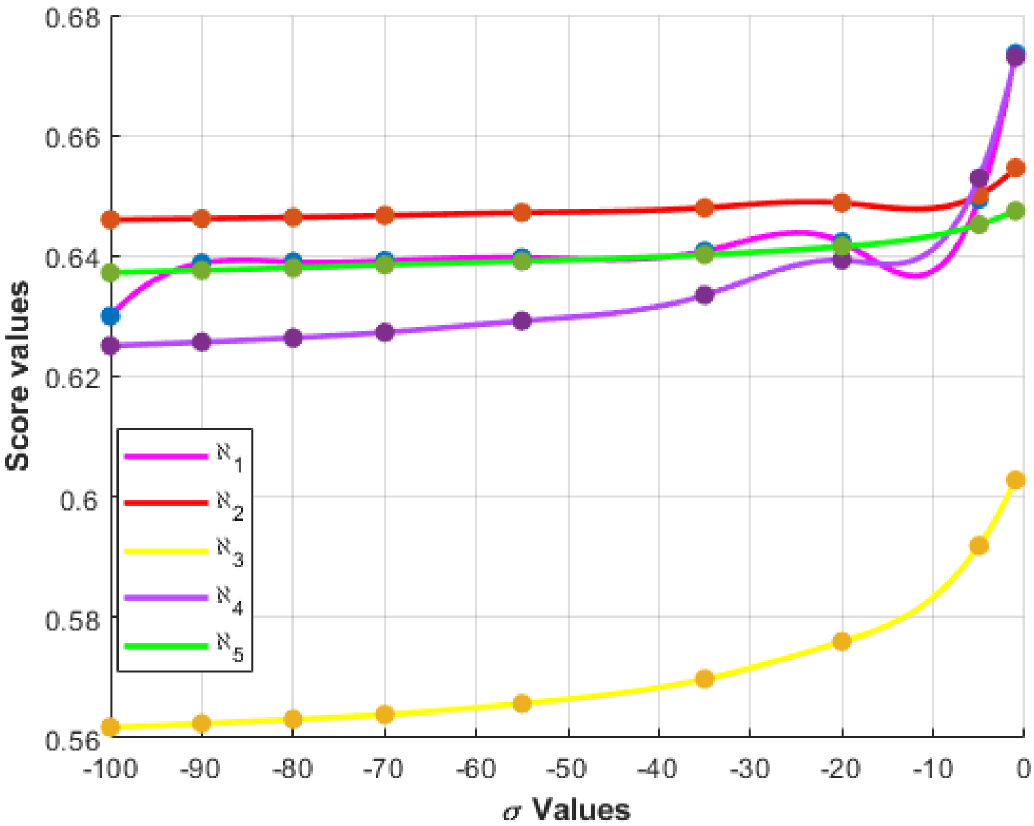

Fig. 2. Impact of σ under SFSSWPG operator

6.2 Comparative Analysis

This segment presents a comparative analysis of the planned approach against existing aggregation-based approaches utilizing various AOs. Ashraf *et al.,* [14] familiarized the spherical fuzzy Dombi weighted averaging (SFDWA) and spherical fuzzy Dombi weighted geometric (SFDWG) operators. Zhang *et al.,* [16] proposed the spherical fuzzy Hamacher power weighted averaging (SFHPWA) and spherical fuzzy Hamacher power weighted geometric (SFHPWG) operators. Hussain *et al.,* [22] established a MADM framework based on the Pythagorean fuzzy rough Schweizer-Sklar weighted averaging (PyFRSSWA) and Pythagorean fuzzy rough Schweizer-Sklar weighted geometric (PyFRSSWG) operators. Furthermore, Jana *et al.,* [40] investigated the application of the picture fuzzy Dombi weighted averaging (PFDWA) and picture fuzzy Dombi weighted geometric (PFDWG) operators for solving MADM problems. We solved the previously mentioned problem by utilizing the aforementioned approaches, and the results are displayed in Table 8.

An analysis of Table 8 discloses that the ranking outcomes attained using the PyFRSSWA operator [22], the SFHPWA operator [16], and the SFDWA operator [14] are identical to those produced by the anticipated SFSSPWA operators. Additionally, the ranking outcomes obtained via the PyFRSSWG operator [22], SFHPWG operator [16], and SFDWA operator [14] exhibit similarity. Yet the results produced by the proposed SFSSWA and SFSSPWG operators are entirely changed.

Table 8
Comparison analysis with some prevailing methods

Operators	*Score values*					Ranking outcomes
	$\aleph_1$	$\aleph_2$	$\aleph_3$	$\aleph_4$	$\aleph_5$	
SFSSPWA	0.7288	0.6105	0.7159	0.6028	0.7288	$\aleph_3 \succcurlyeq \aleph_2 \succcurlyeq \aleph_1 \succcurlyeq \aleph_4 \succcurlyeq \aleph_5$
SFSSPWG	0.6988	0.6687	0.7024	0.6730	0.6988	$\aleph_1 \succcurlyeq \aleph_4 \succcurlyeq \aleph_2 \succcurlyeq \aleph_5 \succcurlyeq \aleph_3$
PyFRSSWA [22]	0.0696	0.0814	0.0956	0.0167	0.0135	$\aleph_3 \succcurlyeq \aleph_2 \succcurlyeq \aleph_1 \succcurlyeq \aleph_4 \succcurlyeq \aleph_5$
PyFRSSWG [22]	0.0607	0.0932	0.0262	0.0456	0.0630	$\aleph_2 \succcurlyeq \aleph_1 \succcurlyeq \aleph_5 \succcurlyeq \aleph_4 \succcurlyeq \aleph_3$
SFHPWA [16]	0.0564	0.0736	0.0939	0.0437	0.0376	$\aleph_3 \succcurlyeq \aleph_2 \succcurlyeq \aleph_1 \succcurlyeq \aleph_4 \succcurlyeq \aleph_5$
SFHPWG [16]	0.0678	0.0866	0.0519	0.0534	0.0647	$\aleph_2 \succcurlyeq \aleph_1 \succcurlyeq \aleph_5 \succcurlyeq \aleph_4 \succcurlyeq \aleph_3$
SFDWA [14]	0.0785	0.0876	0.0975	0.0721	0.0566	$\aleph_3 \succcurlyeq \aleph_2 \succcurlyeq \aleph_1 \succcurlyeq \aleph_4 \succcurlyeq \aleph_5$
SFDWG [14]	0.0672	0.0978	0.0524	0.0148	0.0564	$\aleph_2 \succcurlyeq \aleph_1 \succcurlyeq \aleph_5 \succcurlyeq \aleph_4 \succcurlyeq \aleph_3$
PFDWA [40]	…	…	…	…	…	Unable to determine
PFDWG [40]	…	…	…	…	…	Unable to determine

7. Conclusions

This conclusive investigation successfully translates advanced spherical fuzzy logic into a pragmatic urban infrastructure procurement tool. The proposed robust framework equips city administrators with a defensible, transparent MADM protocol, fundamentally addressing the vagueness inherent in network hardware tenders. By embedding prioritized criteria and Schweizer-Sklar flexibility, it ensures resilience against subjective biases. Future extensions will integrate real-time network telemetry and dynamic weight tuning, ultimately enabling adaptive, data-driven re-evaluations that mirror the evolving demands of smart city digital ecosystems. PAOs are valuable tools for dealing with uncertain information. Meanwhile, the SFS model serves as a robust framework for handling a broad spectrum of uncertain information within decision-analytic paradigms, incorporating membership, abstinence, and non-membership grades. In this study, we introduce an innovative MADM approach based on some new AOs, capitalizing on the versatility and robustness of Schweizer-Sklar operations, PAOs, and SFSs. The key contributions of this research are summarized as follows:

1. We extend the Schweizer-Sklar operations within the framework of the SFS mechanism and develop a class of novel AOs, namely the SFSSPA, SFSSWPA, SFSSPG, and SFSSWPG operators. The key benefit of our created operators over existing ones is that they consider attributes prioritizing into account.
2. The essential properties of these newly projected AOs, including idempotency, boundedness, and monotonicity, are rigorously investigated, along with a detailed examination of their particular cases.
3. Building upon the proposed AOs, we develop a novel MADM methodology within the SFS framework to facilitate more effective alternative ranking.
4. Furthermore, a practical example of optimizing internet router performance is provided to demonstrate the applicability of the established MADM scheme.

5. To validate the robustness of the obtained outcomes, a comprehensive sensitivity analysis is conducted to check the impact of the Schweizer-Sklar parameter on decision outcomes.
6. To demonstrate the significance of the proposed decision-making, a comprehensive comparative analysis is conducted against several existing approaches. Besides, an SRCC analysis is also performed to further evaluate its reliability and effectiveness.

To widen the application spectrum of the planned methodology, future study may explore the following perspectives:

- Extending the anticipated approach to advanced fuzzy paradigms, such as interval-valued SFSs, T-spherical fuzzy sets (T-SFSs) [46], interval-valued T-SFSs, and complex T-SFS, Complex Circular Intuitionistic Fuzzy Dombi sets [44], q-Rung Orthopair Spherical Fuzzy sets [45] to enhance its applicability in more complex decision-making settings with other MADM methoda [46, 48-51].
- Expanding the practical implementation of the projected method across diverse industries, including healthcare, energy, and supply chain management, to deliver valuable insights for real-world applications and facilitate cross-industry comparisons.

Author Contributions

Conceptualization, R.Gul and M.Yousaf; methodology, R.Gul; software, R.Gul; validation, R.Gul and M.Yousaf ; formal analysis, R.Gul; investigation, R.Gul; resources, R.Gul; data curation, R.Gul; writing—original draft preparation, R.Gul; writing—review and editing, R.Gul; visualization, R.Gul; supervision, R.Gul; project administration, X.X.; funding acquisition, R.Gul. All authors have read and agreed to the published version of the manuscript.

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Data Availability Statement

Not applicable.

Conflicts of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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